

Classical and Quantum Mechanics with Poincaré-Snyder Relativity

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Abstract

The Poincaré-Snyder relativity was introduced in an earlier paper of ours as an extended form of Einstein relativity obtained by appropriate limiting setting of the full Quantum Relativity. The latter, with fundamental constants \hbar and G built into the symmetry, is supposed to be the relativity of quantum space-time. Studying the mechanics of Poincaré-Snyder relativity is an important means to get to confront the great challenge of constructing the dynamics of Quantum Relativity. The mechanics will also be of interest on its own, plausibly yielding prediction accessible to experiments. We write the straightforward canonical formulation here, and show that it yields sensible physics pictures. Besides the free particle case, we also give an explicit analysis of two particle collision as dictated by the formulation, as well as the case of a particle rebounding from an insurmountable potential barrier in the time direction. The very interesting solution of particle-antiparticle creation and annihilation as interpreted in a the usual time evolution picture can be obtained, in the simple classical mechanics setting. We consider that a nontrivial success of the theory, giving confidence that the whole background approach is sensible and plausibly on the right track. We also sketch the quantum mechanics formulation, direct from the familiar canoincal quantization, matching to a relativity group geometric quantization formulation in a previous publication.

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I. INTRODUCTION

Poincaré-Snyder Relativity is a relativity we introduced recently [1] as a relativity in between the Galilean or Einstein relativities and the Quantum Relativity as formulated in Ref.[2]. The idea that Einstein relativity has to be modified to admit an invariant quantum scale was discussed by Snyder in 1947[3]. Since the turn of the century, there has been quite some interest in topic [4]. The Quantum Relativity as presented in Ref.[2] has three basic features: 1/ incorporation of fundamental invariant through Lie algebra stabilization [5], 2/ linear realization of new relativity symmetry [6], 3/ the quantum nature to be built in through two invariants — essentially independent Planck momentum and Planck length [2, 7]. Lorentz or Poincaré symmetry (of Einstein relativity) can be considered exactly a result of the stabilization of the Galilean relativity symmetry. The linear realization scheme in that setting is nothing other than the Minkowski space-time picture. Ref.[2] arrived at $SO(2,4)$ as the symmetry for Quantum Relativity or the relativity symmetry for the ‘quantum space-time’ to be realized as a classical geometry of a space-like ‘ AdS_5 ’. Such a mathematically conservative approach leads to a very radical physics perspective. The ‘time’ of Minkowski space-time is not just an extra spatial dimension. Its nature is dictated, from the symmetry stabilization perspective, by the physics of having the invariant speed of light c . The other two new coordinates in our ‘quantum space-time’ picture are likewise dictated. They are neither space nor time [2, 6]. The most important task at hand is then to understand the nature of the new coordinates and their roles in physics. It is with the latter in mind that we introduced the Poincaré-Snyder Relativity in Ref.[1].

In this paper, we want to write down the basics of the canonically formulated mechanics under the Poincaré-Snyder Relativity. The relativity has the symmetry of $G(1,3)$, the mathematical structure of which is like a ‘Galilei group’ in $1+3$ (space-time) dimensions [8]. Table 1 shows the mathematical relation among the various relativity symmetries. Note that the Poincaré-Snyder Relativity is just the Einstein Relativity extended with a new set of transformations — the momentum boosts, without involving the quantum invariants. Poincaré-Snyder mechanics is hence expected to have a mathematical formulation very similar to that under the Galilean or Einstein frameworks. The Poincaré-Snyder Relativity is still a relativity on the 4D Minkowski space-time. However, its mechanics is to be parameterized by the new independent variable σ , the mathematical analog of the time t in the

TABLE I: The various relativities – matching the generators : The table matches out the generators for the various relativity symmetries from a pure mathematical point of view. Note as algebras, the mathematical structures of translations (denoted by P_i) or the boosts (denoted by K_i and K'_i – the so-called Lorentz boosts not included as they are really space-time rotations) in relation to rotations J_i are the same. Algebraically, translation and boost generators are distinguished only by the commutation with the Hamiltonian (H and H'). Successive contractions retrieve $G(1,3)$ and $ISO(1,4)$ from $SO(2,4)$, similar to the more familiar $G(3)$ and $ISO(1,3)$ from $SO(1,4)$. In the physics picture under discussion, however, $SO(1,4)$ part of our so-called Snyder relativity $ISO(1,4)$ is *different* from the usual de-Sitter $SO(1,4)$ contracting to $ISO(1,3)$. We consider simply keeping only the P_μ and $J_{\mu\nu}$ generators to reduce from our Poincaré-Snyder $G(1,3)$ to the Einstein $ISO(1,3)$. More details in Ref.[1].

Relativity Symmetry	Quantum	Snyder	Poincaré-Snyder	Einstein	Galilean
	$SO(2,4)$	$ISO(1,4)$	$G(1,3)$	$ISO(1,3)$	$G(3)$
Arena	‘AdS ₅ ’	M^5	M^4 (with σ)	M^4	\mathbb{R}^3 (with t)
SO(1,4) part	J_{ij}	J_{ij}	J_{ij}	J_{ij}	J_{ij}
	J_{i0}	J_{i0}	J_{i0}	J_{i0}	K_i
	J_{40}	J_{40}	K'_0	P_0	H
	J_{4i}	J_{4i}	K'_i	P_i	P_i
	J_{50}	P_0	P_0		
	J_{5i}	P_i	P_i		
	J_{54}	P_4	H'		

Galilean setting. As the physics meaning of the variable, with physical dimension of $\frac{time}{mass}$, generally differs from time, we will use terms like σ -mechanics and σ -evolution to mark that. It should be taken as a caution sign against standard dynamical interpretations. The momentum boosts are like translations by $p^\mu \sigma$, where the momentum p^μ has been freed from being the standard mass times velocity [6] valid in the Galilean and Einstein frameworks.¹

¹ Although Ref.[6] discusses a candidate relativity symmetry different from the $SO(2,4)$ of Ref.[1] which we follow here, the $ISO(2,4)$ part with the momentum boosts we focus on here is common to both. The

Note that the Einstein rest mass as the magnitude for the energy-momentum four-vector is not an invariant under the momentum boost transformations.

When restricted to the setting of Einstein relativity, Poincaré-Snyder mechanics gives a natural covariant formulation for Einstein mechanics. The same situation holds after quantization. In fact, as the $G(1, 3)$ symmetry admits a nontrivial $U(1)$ central extension, it gives a better framework for quantization[1]. We also address some aspects of quantization for Poincaré-Snyder mechanics in this paper.

Explicitly, the algebra of Poincaré-Snyder symmetry is given by

$$\begin{aligned}
[J_{\mu\nu}, J_{\lambda\rho}] &= -(\eta_{\nu\lambda}J_{\mu\rho} - \eta_{\mu\lambda}J_{\nu\rho} + \eta_{\mu\rho}J_{\nu\lambda} - \eta_{\nu\rho}J_{\mu\lambda}) , \\
[J_{\mu\nu}, P_\rho] &= -(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu) , \\
[J_{\mu\nu}, K'_\rho] &= -(\eta_{\nu\rho}K'_\mu - \eta_{\mu\rho}K'_\nu) , \\
[K'_\mu, H'] &= P_\mu ,
\end{aligned} \tag{1}$$

where the unlisted commutators are all zero and $\eta_{\mu\nu} = (-1, +1, +1, +1)$ ². To the above, we add the commutator

$$[K'_\mu, P_\nu] = \eta_{\mu\nu}F , \tag{2}$$

characterizing the $U(1)$ central extension needed for quantum mechanics [1, 9]. The mathematical symmetry had actually been used as a trick to implement quantization from the latter perspective [10]. Since publishing Ref.[1], we have also come to realized that a relativity symmetry of essentially the same mathematical structure has actually been suggested before. Aghassi, Roman, and Santilli in 1970 argued that (Einstein) relativistic quantum mechanics would better be formulated with an extended relativity/dynamic symmetry duplicating the structure of the Galilei group [11]. That is the $G(1, 3)$ group, which we arrived at from the Quantum Relativity perspective. Though the physics picture of the extra symmetry transformations is different, it is interesting to see that in a sense the top-down approach and the bottom-up approach arise at the same conclusion. We want to emphasize that our basic perspective gives the radical physics picture of being a non-space-time coordinate. The feature set the Poincaré-Snyder physics we try to formulate apart from all earlier attempts

¹ feature is retained in the contraction $G(1, 3)$.

² Note that we follow the metric sign convention adopted in Ref.[1] which is different from that of Refs.[2, 6, 7].

based on similar mathematics. Moreover, the identification of the extra transformations as momentum boosts gives

$$p^\mu := \frac{dx^\mu}{d\sigma} , \quad (3)$$

which is not to be expected from the other physics frameworks. In fact, the equation should be taken as a defining one [2, 6], enforcing the compatibility of which with the standard canonical formulation has nontrivial implications. Note further that σ should not be simply taken as Einstein proper time divides by particle rest mass either. While the latter should hold at the Einstein limit, it has to be relaxed to take the physics of the Poincaré-Snyder Relativity or the Quantum Relativity seriously. In fact, at the Quantum Relativity or Snyder Relativity level, σ as an extra geometric dimension to space-time has a space-like, rather than time-like, signature.³

As an arena for the realization of the relativity symmetry, the Minkowski space-time supplemented by the (absolute) external parameter σ has the transformation (in the passive point of view)

$$x'^\mu = \Lambda^\mu_\nu x^\nu - p^\mu \sigma - A^\mu \quad (4)$$

and σ translation $\sigma' = \sigma - b$. The symmetry action is characterized by the following realization of the generators :

$$\begin{aligned} J_{\mu\nu} &\rightarrow - \left(x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu} \right) , & K'_\mu &\rightarrow -\sigma \frac{\partial}{\partial x^\mu} , \\ P_\mu &\rightarrow -\frac{\partial}{\partial x^\mu} , & H' &\rightarrow -\frac{\partial}{\partial \sigma} . \end{aligned} \quad (5)$$

Similar to the Galilean case, a canonical realization [12] is admissible on an (one particle) phase space with the introduction of the momentum variables conjugate to x^μ . The latter are just p_μ . We illustrate the explicit canonical formulation here, demonstrating in particular the consistency with the ‘new’ momentum definition of Eq.(3).

³ Though Ref.[11] starts by formulating the $G(1,3)$ as a new ‘dynamical’ symmetry, it assumes the ‘evolution’ parameter σ (as we discuss in the section below) as essentially a time parameter. That is equivalent to keeping momentum as mass times velocity hence forcing the otherwise independent coordinate/parameter σ to be essentially the Einstein proper time. It actually implies the K'_i generators are *not* independent of the Lorentz boosts J_{i0} . Hence, the role of the $G(1,3)$ symmetry in physics within the framework is really not much different from that of Ref.[9]. In our approach, $G(1,3)$ is actually a basic kinematic symmetry with an independent σ parameter playing a fundamental role. We will discuss the issue further after we elaborate our formulation of the σ -mechanics.

As discussed above and in Ref.[1], a straightforward canonical Lagrangian/Hamiltonian formulation for Poincaré-Snyder mechanics looks admissible and sensible, at least in the mathematical sense. One has to be more cautious with the physics interpretation. In this paper, we write down the formulation explicitly and look at a couple of simple cases, namely free particle dynamics and the collisional dynamics of two otherwise free particles. We take caution against committing to any definite physics interpretation of the σ parameter and the new momentum boost transformations, including any possible practical implementation of the latter. With the minimalist approach, we show the results can be sensibly interpreted in a usual time evolution picture. We focus on classical mechanics, elaborated in the next section. Quantization of the classical mechanics will then be addressed in sec.III, after which we conclude in the last section.

II. CANONICAL FORMULATION

The mathematics of the canonical formulation of the σ -mechanics is straightforward. We mostly want to write down things explicitly for future usage, and to elaborate, whenever possible, on the physics picture that sets it apart from systems with similar mathematics. The most important point to note here is that time t , or $x^0 = ct$ to be exact, is now put on the same footing as the spacial position x^i , while we have a new absolute ‘evolution’ parameter σ . However, σ only parameterizes the ‘evolution’ of the system in a formal sense. It is generally speaking not a sort of time parameter. We highlight the feature by terms like σ -evolution and σ -Lagrangian.

A. σ -Lagrangian and σ -Hamiltonian

We start with a variational principle on the σ -Lagrangian, for the simple ‘one particle’ system with configuration variables x^μ as a prototype and specific case of interest. The σ -Lagrangian would be a function $\mathcal{L}(x^\mu, \dot{x}^\mu, \sigma)$, where the ‘dot’ here denotes differentiation with respect to the ‘evolution’ parameter σ , *i.e.* $\dot{x}^\mu \equiv \frac{dx^\mu}{d\sigma}$. However, that is just p^μ [cf. Eq.(3)]; hence

$$\mathcal{L}(x^\mu, \dot{x}^\mu, \sigma) = \mathcal{L}(x^\mu, p^\mu, \sigma) . \quad (6)$$

The above is specific, only for the case with x^μ as configuration variables. The standard procedure introduces the canonical momentum through $p_\mu = \frac{\partial \mathcal{L}(x^\mu, \dot{x}^\mu, \sigma)}{\partial \dot{x}^\mu}$, which can now be written as

$$p_\mu = \frac{\partial \mathcal{L}(x^\mu, p^\mu, \sigma)}{\partial p^\mu} . \quad (7)$$

A Legendre transformation gives the σ -Hamiltonian as

$$\mathcal{H}(x^\mu, p_\mu, \sigma) = p_\mu \dot{x}^\mu - \mathcal{L}(x^\mu, \dot{x}^\mu, \sigma) = p_\mu p^\mu - \mathcal{L}(x^\mu, p^\mu, \sigma) . \quad (8)$$

The action functional on the phase space $\{x^\mu, p_\mu\}$ can be written as $S[\gamma] \equiv \int_{\sigma'}^{\sigma''} d\sigma [p_\mu \dot{x}^\mu - \mathcal{H}(x^\mu, p_\mu, \sigma)]$, where γ denote a phase space trajectory parameterized by σ .

Variations with respect to x^μ and p_μ give us:

$$\delta S[\gamma] = \int_{\sigma'}^{\sigma''} d\sigma \left[\frac{d(\delta x^\mu p_\mu)}{d\sigma} + \delta p_\mu \dot{x}^\mu - \delta x^\mu \frac{dp_\mu}{d\sigma} - \frac{\partial \mathcal{H}(x^\mu, p_\mu, \sigma)}{\partial x^\mu} \delta x^\mu - \frac{\partial \mathcal{H}(x^\mu, p_\mu, \sigma)}{\partial p_\mu} \delta p_\mu \right] . \quad (9)$$

Hence, the principle of stationary action yields the Hamiltonian equation of motion

$$\begin{aligned} \frac{dx^\mu}{d\sigma} &= \frac{\partial \mathcal{H}(x^\mu, p_\mu, \sigma)}{\partial p_\mu} , \\ \frac{dp_\mu}{d\sigma} &= - \frac{\partial \mathcal{H}(x^\mu, p_\mu, \sigma)}{\partial x^\mu} , \end{aligned} \quad (10)$$

which implies

$$\frac{d\mathcal{H}}{d\sigma} = \frac{\partial \mathcal{H}(x^\mu, p_\mu, \sigma)}{\partial \sigma} . \quad (11)$$

Of course one expect the formal structure of the Lagrangian and Hamiltonian formulation to maintain when being applied to current framework. The formulations are not restricted to a specific relativity, though their application to the Einstein relativistic system is often written in a constrained form [13]. For a generic set of canonical variables, we do not necessarily have a relation between the canonical configuration and momentum variables as given by Eq.(3) and the σ -evolution has only the usual formal description.

Let us elaborate further on the ‘one particle’ system, which has specific features that will also apply to system with space-time locations as configuration variables. Altogether, it is easy to see that our special definition in Eq.(3) requires

$$\begin{aligned} \mathcal{L}(x^\mu, p^\mu, \sigma) &= \frac{1}{2} \eta_{\mu\nu} p^\mu p^\nu - \Phi(x^\mu, \sigma) , \\ \mathcal{H}(x^\mu, p_\mu, \sigma) &= \frac{1}{2} \eta^{\mu\nu} p_\mu p_\nu + \Phi(x^\mu, \sigma) . \end{aligned} \quad (12)$$

The essentially $\frac{1}{2}p_\mu p^\mu$ term, is to be compared to the $\frac{1}{2}mv^2 = \frac{p^2}{2m}$ kinetic energy term common to the Lagrangian and the Hamiltonian for the case of Galilean relativity. In the latter case, it is enforced by the particle momentum definition of $p^i := mv^i$. It is interesting to note that there is no free parameter in the term $\frac{1}{2}p_\mu p^\mu$. There is no explicit role in the formulation for any particle property like mass. However, one will see in our discussion of the free particle, *i.e.* $\Phi = 0$, case below that the term particle is still justified in the framework.

B. Symplectic Geometric Description

The phase space for the one-particle system bears the standard symplectic structure as the cotangent bundle of the Minkowski space-time. The symplectic 2-form is $\omega = dp_\mu \wedge dx^\mu$, which obviously generalizes directly to an N -particle system. We adopt the notation $\{q^a, p_a\}$ for a set of canonical coordinates of a generic Poincaré-Snyder system. The Poincaré form on $T^*M \times \mathbb{R}$, where M is the configuration space coordinated by $\{q^i\}$, has the standard form $\Lambda = p_a dq^a - \mathcal{H} d\sigma$ with \mathcal{H} as a σ -Hamiltonian. The Hamiltonian equations of motion (II A) and (11) are summarized in

$$i_{X_{\mathcal{H}}} \Omega = -d\mathcal{H} ,$$

where $\Omega = d\Lambda = dp_a \wedge dq^a - d\mathcal{H} \wedge d\sigma$; $X_{\mathcal{H}}$ is the Hamiltonian vector field. The action functional $S[\gamma]$ is geometrically $\int_\gamma \Lambda$.

For a σ independent \mathcal{H} , we have

$$i_{X_{\mathcal{H}}} \omega = -d\mathcal{H} ,$$

where $\omega = dp_a \wedge dq^a$. for the function on the cotangent bundle \mathcal{H} as our σ -Hamiltonian. Poisson bracket for functions f and g is given by

$$-\omega(X_f, X_g) \equiv \{f, g\} = \sum_a \left(\frac{\partial f}{\partial q^a} \frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a} \frac{\partial g}{\partial q^a} \right) , \quad (13)$$

with $X_{\mathcal{H}} = \{\cdot, \mathcal{H}\}$. In terms of the Poisson bracket, we have

$$\frac{df}{d\sigma} = \{f, \mathcal{H}\} + \frac{\partial f}{\partial \sigma} . \quad (14)$$

which summaries the mechanics of σ -(Hamiltonian) evolution.

Everything given in this sub-section is standard in form.

C. Canonical Transformations and Lorentz Transformations

There are several equivalent definitions of canonical transformations [13], here we follow the most common one, the so-called generating function approach. After writing down the general theory of canonical transformation, we present generating functions correspond to the transformations of Poincaré-Snyder Relativity symmetry itself. The latter is the main focus of our presentation here. We also study infinitesimal version of such generating functions, whenever they leave the σ -Hamiltonian function invariant, which are constant of motions with respect to σ -evolution. A direct advantage of studying canonical transformations in the context of Poincaré-Snyder relativity is that the Lorentz transformations can be easily incorporated into the framework.

We define transformations from phase space variables (q^a, p_a) to (Q^a, P_a) being canonical if they preserve the structure of Hamiltonian equation of motion. That is

$$\frac{dq^a}{d\sigma} = \frac{\partial \mathcal{H}(q^a, p_a, \sigma)}{\partial p_a}, \quad \frac{dp_a}{d\sigma} = -\frac{\partial \mathcal{H}(q^a, p_a, \sigma)}{\partial q^a}, \quad \frac{d\mathcal{H}}{d\sigma} = \frac{\partial \mathcal{H}(q^a, p_a, \sigma)}{\partial \sigma}, \quad (15)$$

if and only if

$$\frac{dQ^a}{d\sigma} = \frac{\partial \mathcal{K}(Q^a, P_a, \sigma)}{\partial P_a}, \quad \frac{dP_a}{d\sigma} = -\frac{\partial \mathcal{K}(Q^a, P_a, \sigma)}{\partial Q^a}, \quad \frac{d\mathcal{K}}{d\sigma} = \frac{\partial \mathcal{K}(Q^a, P_a, \sigma)}{\partial \sigma}, \quad (16)$$

where $\mathcal{K}(Q^a, P_a)$ is the same σ -Hamiltonian function represented in the new phase space variable (Q^a, P_a, σ) . Its functional form is in general different from $\mathcal{H}(q^a, p_a, \sigma)$. To find a systematic way to generate such transformations, we recall that the Hamiltonian equation of motion is derived from phase space variational principle $\delta \int [p_a \dot{q}^a - \mathcal{H}(q^a, p_a, \sigma)] d\sigma = 0$, treating δp_a and δq^a independently. The special form of the integrand is the essence that leads to the canonical form of equation of motion, therefore in the (Q^a, P_a) coordinates we must also have $\delta \int P_a \dot{Q}^a - \mathcal{K}(Q^a, P_a, \sigma) = 0$. It is not difficult to see the condition for above two variations hold is that

$$P_a \dot{Q}^a - \mathcal{K}(Q^a, P_a, \sigma) = p_a \dot{q}^a - \mathcal{H}(q^a, p_a, \sigma) + \frac{dF(q^a, Q^a, p_a, P_a)}{d\sigma}, \quad (17)$$

since we require the variations at end points to be zero. The variables of function F is not all independent, they are related by the transformations

$$Q^a = Q^a(q^a, p_a, \sigma), \quad P_a = P_a(q^a, p_a, \sigma). \quad (18)$$

Only half of them are independent variables. For our convenience here, we take $\{q^a, P_a\}$ as independent variables and $F(q^a, P_a, \sigma)$ of the particular form

$$F = F_2(q^a, P_a, \sigma) - Q^a P_a. \quad (19)$$

Substitute this equation into Eq.(17), we obtain the following relations:

$$\begin{aligned} p_a &= \frac{\partial F_2}{\partial q^a}(q^a, P_a, \sigma), & Q^a &= \frac{\partial F_2}{\partial P_a}(q^a, P_a, \sigma), \\ \mathcal{K}(Q^a, P_a, \sigma) &= \mathcal{H}(q^a, p_a, \sigma) + \frac{\partial F_2}{\partial \sigma}(q^a, P_a, \sigma). \end{aligned} \quad (20)$$

The equations give an implicit relation between the two sets of canonical variables, from which one can obtain the explicit transformation $(q^a, p_a) \rightarrow (Q^a, P_a)$, and the inverse transformation, so long as the invertibility condition for set of equations

$$\det \left(\frac{\partial^2 F_2}{\partial P_b \partial q^a} \right) \neq 0$$

is satisfied.

With the above sketch of formulation of canonical transformations, we are ready to present generating functions that correspond to the transformations of the Poincaré-Snyder Relativity symmetry. We first identify (q^a, p_a) and (Q^a, P_a) as (x^μ, p_μ) and (x'^μ, p'_μ) , respectively.

1./ identity transformation —

The generating function for the identity transformation $x'^\mu = x^\mu$, $p'^\mu = p^\mu$ can obviously be given by

$$F_{id}(x^\mu, p'_\mu) = x^\mu p'_\mu. \quad (21)$$

2./ space-time translations —

A space-time translation characterized by a finite four vector \mathcal{A}^μ can be generated by

$$F_P(x^\mu, p'_\mu) = x^\mu p'_\mu - p'_\mu \mathcal{A}^\mu, \quad (22)$$

giving explicitly

$$p_\mu = \frac{\partial F_P}{\partial x^\mu} = p'_\mu, \quad x'^\mu = \frac{\partial F_P}{\partial p'_\mu} = x^\mu - \mathcal{A}^\mu. \quad (23)$$

The associated transformation of momentum is what it has to be, as the last equation implies $dx'^\mu = dx^\mu$ giving $p'^\mu = \frac{dx'^\mu}{d\sigma} = \frac{dx^\mu}{d\sigma} = p^\mu$. Of course F_P reduces to F_{id} when $\mathcal{A}^\mu = 0$.

3./ momentum boosts —

Since momentum boosts in Poincaré-Snyder Relativity are characterized by a constant four

momentum \mathcal{P}^μ and a particular value of the evolution parameter σ , its generating function now must also be σ dependent. The generating function and its partial derivatives reads

$$\begin{aligned} F_{K'}(x^\mu, p'_\mu, \sigma) &= (x^\mu - \mathcal{P}^\mu \sigma) p'_\mu + \mathcal{P}_\mu x^\mu + f(\sigma), \quad f(\sigma) = -\frac{1}{2} \mathcal{P}_\mu \mathcal{P}^\mu \sigma, \\ p_\mu &= \frac{\partial F_{K'}}{\partial x^\mu} = p'_\mu + \mathcal{P}_\mu, \quad x'^\mu = \frac{\partial F_{K'}}{\partial p'_\mu} = x^\mu - \mathcal{P}^\mu \sigma. \end{aligned} \quad (24)$$

Note that $dx'^\mu = dx^\mu - \mathcal{P}^\mu d\sigma$, hence the consistency of the transformation for the momentum variables. The generating function reduces to F_{id} under the condition that either $\sigma = 0$ or $\mathcal{P}^\mu = 0$. Strictly speaking the function $f(\sigma)$ is arbitrary in the formulation. Note that its choice does not affect the transformation among the canonical variable as given by the second line of equations. The choice of $-\frac{1}{2} \mathcal{P}_\mu \mathcal{P}^\mu \sigma$ here is what would leaves the Hamiltonian exactly invariant:

$$\mathcal{H}(x'^\mu, p'_\mu) = \frac{1}{2} p_\mu p^\mu + \frac{d}{d\sigma} \left(-\frac{1}{2} \mathcal{P}_\mu \mathcal{P}^\mu \sigma - p'_\mu \mathcal{P}^\mu \sigma \right) = \mathcal{H}(x^\mu, p_\mu). \quad (25)$$

The momentum boosts differ from the other transformations in the symmetry for as transformations on the space-time, of $x'^\mu = x^\mu - \mathcal{P}^\mu \sigma$, leaves the Lagrangian or the Hamiltonian only quasi-invariant. The special feature has consequences we will discuss in a few places below.

4./space-time rotations (Lorentz transformations) —

Space-time rotations includes the special cases of 3 dimensional rotations, pure Lorentz boosts, and the mixture of them. The generating function and its partial derivatives reads

$$F_J(x^\mu, p'_\mu) = p'_\rho \Lambda^\rho_\lambda x^\lambda, \quad p_\mu = \frac{\partial F_J}{\partial x^\mu} = p'_\rho \Lambda^\rho_\mu, \quad x'^\mu = \frac{\partial F_J}{\partial p'_\mu} = \Lambda^\mu_\lambda x^\lambda. \quad (26)$$

Again, the transformation rule for momentum obtained is consistent with the one obtained from its definition, namely $p'^\mu = \frac{dx'^\mu}{d\sigma} = \Lambda^\mu_\nu \frac{dx^\nu}{d\sigma} = \Lambda^\mu_\nu p^\nu$. In particular, the generating function for, explicitly, a spatial rotation around the z -axis (through angle θ) and that for a pure Lorentz boost along x -axis (by velocity v) are given, respectively, by

$$F_{\mathbf{z}} = p'_0 x^0 + \cos\theta (p'_1 x^1 + p'_2 x^2) + p'_3 x^3 - \sin\theta (x^1 p'_2 - x^2 p'_1), \quad (27)$$

$$F_{\mathbf{0}} = \gamma(p'_0 x^0 + p'_1 x^1) + p'_2 x^2 + p'_3 x^3 - \gamma \frac{v}{c} (x^0 p'_1 + x^1 p'_0), \quad (28)$$

where $\gamma = 1/\sqrt{1 - (\frac{v}{c})^2}$.

Infinitesimal canonical transformations have generators $G(q^a, P_a, \sigma)$ satisfying

$$F_2(q^a, P_a, \sigma) = q^a P_a + \epsilon G(q^a, P_a, \sigma), \quad (29)$$

with ϵ being an infinitesimal parameter. From Eq.(20), we obtain

$$\mathcal{H}\left(q^a + \epsilon \frac{\partial G}{\partial p_a}, p_a - \epsilon \frac{\partial G}{\partial q^a}, \sigma\right) = \mathcal{H}(q^a, p_a, \sigma) + \epsilon \frac{\partial G}{\partial \sigma}. \quad (30)$$

Hence, to leave the σ -Hamiltonian invariant, *i.e.* $\mathcal{K}(Q^a, P_a, \sigma) = \mathcal{H}(Q^a, P_a, \sigma)$, it is required that

$$\frac{dG}{d\sigma} = \{G, \mathcal{H}\} + \frac{\partial G}{\partial \sigma} = 0. \quad (31)$$

The above gives the generators as constants of σ -evolution. For example, for the free particle σ -mechanics with $\mathcal{H} = \frac{1}{2}p^\mu p_\mu$, to be discussed below, generators for the relativity symmetry transformations give the following constants of motion :

$$\begin{aligned} G_{P_\mu} &= -p_\mu & (\text{momentum}) , \\ G_{K'_\nu} &= -p_\mu \sigma + x_\mu & (\text{center of mass}) , \\ G_{J_{\mu\nu}} &= -(x_\mu p_\nu - x_\nu p_\mu) & (\text{angular momentum}) . \end{aligned} \quad (32)$$

Of course the σ -Hamiltonian is itself $G_{H'}$. Note that Poisson brackets among the generators give a realization of the original algebra extended with a central charge with the value of unity ⁴, *i.e.*

$$\begin{aligned} \{G_{J_{\mu\nu}}, G_{J_{\lambda\rho}}\} &= \eta_{\nu\lambda} G_{J_{\mu\rho}} - \eta_{\mu\lambda} G_{J_{\nu\rho}} + \eta_{\mu\rho} G_{J_{\nu\lambda}} - \eta_{\nu\rho} G_{J_{\mu\lambda}} , \\ \{G_{J_{\mu\nu}}, G_{P_\lambda}\} &= \eta_{\nu\lambda} G_{P_\mu} - \eta_{\mu\lambda} G_{P_\nu} , \quad \{G_{J_{\mu\nu}}, G_{K'_\lambda}\} = \eta_{\nu\lambda} G_{K'_\mu} - \eta_{\mu\lambda} G_{K'_\nu} , \\ \{G_{K'_\mu}, G_{P_\nu}\} &= -\{x_\mu, p_\nu\} = -\eta_{\mu\nu} , \quad \{G_{K'_\mu}, G_{H'}\} = -G_{P_\mu} , \\ \{G_{P_\mu}, G_{P_\nu}\} &= \{G_{P_\mu}, G_{H'}\} = \{G_{J_{\mu\nu}}, G_{H'}\} = \{G_{K'_\mu}, G_{K'_\nu}\} = 0 . \end{aligned} \quad (33)$$

[*cf.* Eqs. (1) and (2)]. Explicitly, all Poisson brackets among the generators match to the corresponding commutators with a negative sign illustrating explicitly the formulation as a canonical realization of the Poincaré-Snyder symmetry; hence the central charge as the value for F generator in Eq.(2) is +1. Fixing the central charge as unity is a special feature here compared to the corresponding formulation for the Galilean particle. This will be discussed in the description of free particle mechanics below.

⁴ The $f(\sigma)$ function in Eq.(24) is what allows the restoration of p_μ from p'_μ is $G_{K'}$, the only case in which $p'_\mu \neq p_\mu$ in the infinitesimal limit. The necessity and arbitrariness is related to the quasi-invariant properties of the momentum boosts. The latter is connected with the appearance of the ‘extra’ x_μ in $G_{K'_\nu}$ or the Noether current and the modification of the commutator $[K'_\mu, P_\nu]$ with a central extension — all consequences of the nontrivial cohomology of the $G(1, 3)$ group in exact analog to the Galilean $G(3)$ [1, 9].

D. Hamilton-Jacobi Equation

As usual, generating functions for canonical transformations based on the phase space action lead to Hamilton-Jacobi equation. Consider σ -evolution given by

$$\begin{aligned} Q^a &\equiv q^a(\sigma) = q^a(q^a(0), p_a(0), \sigma_0, \sigma) \\ P^a &\equiv p_a(\sigma) = p_a(q^a(0), p_a(0), \sigma_0, \sigma) \end{aligned}$$

where $q^a(0)$ and $p_a(0)$ denote the ‘initial’ value of q^a and p_a at $\sigma = \sigma_0$. Below within the subsection, we use simply q^a and p_a for the latter. For fixed Q^a and P_a , constant in σ , the generating function $S \equiv F_2(q^a, P_a, \sigma)$ satisfies

$$\mathcal{K}(Q^a, P_a, \sigma) = \mathcal{H}(q^a, p_a, \sigma) + \frac{\partial S}{\partial \sigma} = 0 .$$

We have

$$p_a = \frac{\partial S}{\partial q^a} \quad \text{and} \quad Q^a = \frac{\partial S}{\partial P_a} ,$$

hence

$$\mathcal{H}\left(q^a, \frac{\partial S}{\partial q^a}, \sigma\right) + \frac{\partial S}{\partial \sigma}(q^a, P_a, \sigma) = 0 . \quad (34)$$

That is the Hamilton-Jacobi equation for σ -mechanics.

The above is also standard and straightforward, and again

$$\frac{dS}{d\sigma} = \frac{\partial S}{\partial q^a} \dot{q}^a + \frac{\partial S}{\partial \sigma} = p_a \dot{q}^a - \mathcal{H} \quad (35)$$

illustrating that S is the action re-interpreted as the function $S(q^a, P_a, \sigma)$.

E. Free Particle σ -mechanics as an Einstein Limit

For the free particle σ -Hamiltonian $\mathcal{H}(x^\mu, p_\mu) = \frac{1}{2}p_\mu p^\mu$, σ -mechanics is trivial. The equations of motion, as one expects, are $\frac{dx^\mu}{d\sigma} = p^\mu$, $\frac{dp_\mu}{d\sigma} = 0$, and $\frac{d\mathcal{H}}{d\sigma} = 0$. Note that the first equation is only a necessary result as it is exactly our definition for p^μ [*cf.* Eq.(3)], which is an analog for the velocity definition $v^i = \frac{dx^i}{dt}$ under Galilean relativity. We can write the constant value for \mathcal{H} as $-\frac{1}{2}m^2c^2$ with c being the speed of light while m^2 is here, we emphasize, just a number to characterize the value of \mathcal{H} . The p^μ definition then yields $(m d\sigma)^2 = -\frac{1}{c^2} dx_\mu dx^\mu = (d\tau)^2$, where τ is the familiar Einstein proper time. Most interestingly, the equation implies $p^\mu = \pm m \frac{dx^\mu}{d\tau}$. Taking the positive sign, we have retrieved

the Einstein interpretation of our momentum four-vector provided by the identification of $m = \frac{d\tau}{d\sigma}$ as the rest mass of an Einstein particle, which is feasible at least as long as the value of \mathcal{H} is negative. Restricting to negative values for the σ -Hamiltonian and positive $\frac{d\tau}{d\sigma}$, free particle σ -mechanics is just a reformulation of Einstein relativistic mechanics for a free particle. However, instead of having the constant m as a property characterizing the particle/system, in the σ -mechanics description, m gives only essentially the value of the σ -Hamiltonian. Instead of having particles with different masses, we have only one (free) particle with different \mathcal{H} values. More important is the fact that the concept of an invariant particle mass then should not maintain its meaning in the presence of interactions. Another paramount feature to note is that as Galilean kinetic energy is not invariant under a velocity boost, our $\frac{1}{2}p_\mu p^\mu$ and hence m as a ‘properties’ for the free particle is not invariant under a momentum boost [*cf.* Eq.(24)].

There is another way to see the Einstein nature of the free particle σ -mechanics. Consider the quantity $\mathcal{H}_c(x^\mu, p_\mu) = \frac{1}{2}p_\mu p^\mu + \frac{1}{2}m^2 c^2$ defined to have vanishing value. If we take $\mathcal{H}_c(x^\mu, p_\mu)$ as a Hamiltonian, we expect it to give the same ‘mechanics’ with $\mathcal{H}(x^\mu, p_\mu)$ as the two differ only by a constant, except that $\mathcal{H}_c(x^\mu, p_\mu)$ is a constrained Hamiltonian. Applying the variational principle to $\mathcal{H}_c(x^\mu, p_\mu)$ with the constraint without reference to σ is exactly what is done in the covariant extended Hamiltonian formulation of Einstein relativistic mechanics [13]. The standard approach gives the equations of motion with an undetermined Lagrange multiplier λ , namely $\frac{dx^\mu}{d\beta} = \lambda p^\mu$ where β is a parameter introduced to parameterize the evolution. The identification $d\sigma = \lambda d\beta$ restores results of our σ -mechanics. The constrained formulation requires further $\frac{dt}{d\beta} > 0$, for the normal particle, which gives $p_0 = -\sqrt{p_i p^i + m^2 c^2}$. Note that from $p_\mu p^\mu = -m^2 c^2 < 0$, we do need $-p_0 = p^0 > 0$ to have $p^0 = \gamma mc > 0$ for the normal particle, *i.e.* positive energy.

Naively, the $p^\mu = -m \frac{dx^\mu}{d\tau}$, or equivalently $m = -\frac{d\tau}{d\sigma}$, case differs only by a conventional redefinition of σ as $-\sigma$, hence unimportant. However, as σ here is an ‘absolute’ parameter in analog to Galilean/Newtonian time, we cannot avoid having to deal with a relative negative sign the $\frac{dx^0}{d\tau}$ values of two non-interacting ‘particles’. Hence, a ‘particle’ can have negative value of $m = \frac{d\tau}{d\sigma}$, which means its (proper) time ‘evolves’ in σ backwards or in opposite direction to the familiar Einstein particle. It is an antiparticle, essentially in same sense as the Stückelberg-Feynman idea [14] formulated, in our opinion, in a better conceptual framework. The antiparticle is the forward time evolving interpretation of a free particle (in

σ -mechanics) that evolves/moves backwards in time, for as traced by the absolute/common σ , its (proper) time ‘evolves’ in opposite direction to that of a (normal) forward time evolving particle. When one goes beyond free particle/antiparticle σ -mechanics, it is easy to contemplate $\frac{d\tau}{d\sigma}$ evolution that flips sign at some point, giving something similar to the usual picture of pair creation/annihilation. We will discuss the kind of scenario below in two-particle scattering/collision analysis.

Next we consider a free particle with $\mathcal{H} = 0$. It is quite obvious that the σ -mechanics describes in that case the free motion of a particle with zero rest mass. Of course the particle will be moving with the speed of light when observed from any Lorentz frame; mathematically, $\frac{d\tau}{d\sigma} = 0$ means there is no σ -evolution in its proper time. Remember though the mass value of a particle depends on the choice of reference frames relative to momentum boosts. And we have been suspending the question of if and how one can implement a momentum boost physically.

Finally, one has to consider positive values of \mathcal{H} . Sticking to $\mathcal{H} = -\frac{1}{2}m^2c^2$, we have $m^2 < 0$ and the momentum four-vector is space-like instead of time-like. In the Einstein framework, it looks like a tachyon. Naturally, we have tachyonic particle as well as tachyonic antiparticles admissible. Note that Einstein relativity does not excluded tachyonic mechanics, though there may be difficulties in thinking about interactions among tachyons and normal particles [15]⁵. However, given the problem in describing interacting normal particles (no interaction theorem) [16], it is hard to see the latter as a short-coming of the idea of tachyons being physical. It may rather be a limitation of the Einstein framework itself.

We see that free particle σ -mechanics naturally describes a free Einstein particle of the normal form, as well as the antiparticle, tachyonic particles, and tachyonic antiparticles, all on the same footing. The true exciting story is in the potential to describe interactions among such ‘particles’. While the mathematics of such a description would be straightforward, the physical interpretation is expected to take thinking beyond the familiar conceptual framework, in connection to the physical understanding of the σ parameter itself. To that we want to proceed with upmost caution. A minimal approach is to use only a relational

⁵ Einstein relativity put a clear separation between the normal particle world and the world of tachyonic particles. However, it is important to note that normal particles appear tachyonic to the tachyons, while tachyons appear to each other normal. Velocity is relative, hence also the distinction between normal and tachyonic particles (see Ref.[15] for discussions).

description. That is a solution path $x^\mu(\sigma)$ gives at each σ value the corresponding values for x^i and t hence $x^i(t)$ implicitly. In fact, our laboratory picture of the motion is only to be interpreted as $x^i(t)$, maybe with an extra $\sigma(t)$. We can focus on the former part first, without committing ourselves to the general physical meaning and plausible measurement of σ . A good example of the interpretation is already illustrated by the antiparticle picture, adopted from Stückelberg-Feynman [14]. We will have more examples below.

Poincaré-Snyder free particle has m value characterizing the magnitude of the four-momentum and hence the σ -Hamiltonian. The value is dependent on the choice of reference frame related by momentum boosts. The corresponding formal σ evolution picture (or simply σ -picture) is what our canonical formulation here gives directly. The notions of particle, antiparticle, tachyonic particle, and tachyonic antiparticle, are notions in the Einstein time evolution picture (t -picture) within which one freeze the option of momentum boosts. The latter is the one we use here to interpret results from our σ -mechanics.

F. Particle(s) with Interactions: A First Look

We have seen that in σ -mechanics with nontrivial potential, the magnitude of the four-momentum is no longer constant. Hence, the idea of a particle as a point mass loses its validity. Here in this subsection, we adopt an abuse of terminology and keep the term particle to mean a point objection characterized by a space-time position x^μ or rather called an Einstein event at each value of σ . Within a self-consistent canonical framework, we start to look into simple systems with particle(s) in interaction. Readers should bear in mind that, similar to quantum mechanics, a state of say a two-particle system is expected to resemble one of two Einstein particles (including antiparticles and tachyons) only at the asymptotic limit where the interaction is negligible.

We first illustrate the Stückelberg-Feynman picture of particle-antiparticle annihilation and pair creation. From the above canonical formulation, we have for a general potential $\Phi(x^\mu)$

$$\frac{dp_\mu}{d\sigma} = -\frac{\partial\mathcal{H}}{\partial x^\mu} = -\frac{\partial\Phi}{\partial x^\mu}. \quad (36)$$

If one take a potential dependent only on x^0 , in fact proportional to a the delta-function in x^0 , the pair annihilation picture will be resulted. Let us give a bit of details. From the equation, we see that momentum components in the spacial x^i directions will have constant

values. The time component, however, will not be conserved. So, the energy of an Einstein particle as essentially given by $\frac{dx^0}{d\sigma}$ is not constant in σ . It change value when $\frac{\partial\Phi}{\partial x^0}$ is nonzero, hence only at one value of $t = x^0/c$ (say $t = t_o$). As the σ -Hamiltonian in this case still has no explicit dependence on σ , its value is conserved. Hence, the different but constant values of the $\frac{dx^0}{d\sigma}$ before and after the σ value with $x^0(\sigma = \sigma_o) = ct_o$ have to give the same value for $\frac{1}{2}p^\mu p_\mu$, the value of the free particle σ -Hamiltonian for $\sigma \neq \sigma_o$. Having $p^0 = \frac{dx^0}{d\sigma}$ flip sign is the only solution. One can also think about the particle, with initiate $\frac{dx^0}{d\sigma} > 0$, as confronting an insurmountable potential barrier in the t -direction and bounces back. In the Einstein picture, the sign flip changes the particle into an antiparticle, with p^i directions reversed. When we look at the situation with a time evolution point of view, we get particle-antiparticle annihilation at $t = t_o$, as shown in Fig.1.

A pair creation picture can easily been formulated in a similar fashion. We take the same potential with $\frac{\partial\Phi}{\partial x^0}$ nonzero only at $t = t_o$. All we have to do is to take a σ -particle with initiate $\frac{dx^0}{d\sigma} < 0$ ‘evolving’ towards $\sigma = \sigma_o$ corresponding to $t = t_o$. In the t -picture, it will be seen as an antiparticle emerging from $t = t_o$ with σ decreases in time from the value σ_o . The solution from the σ -evolution for $\sigma > \sigma_o$ will give a $\frac{dx^0}{d\sigma} > 0$ line, hence an Einstein particle also emitted at $t = t_o$, completing the pair creation story. Notice that the pair creation and pair annihilation description here works independent of the magnitude of the initiate or final momentum four-vector or the numerical value of σ -Hamiltonian. Hence, it works also for massless particles such as a pair of photons (photon being its own antiparticle). It is easy to imagine that some explicit description of a realistic potential between two particles at the right initiate condition in the σ -evolution can mediate the pair annihilation and pair creation parts and describe a realistic experimental setting of such instances.

Let us now take our analysis to the two particle case. The σ -Hamiltonian of two interacting particles of the form $\mathcal{H} = \frac{1}{2}p_a^\mu p_{a\mu} + \frac{1}{2}p_b^\mu p_{b\mu} + \Phi(|x_a^\mu - x_b^\mu|)$ can be canonically transformed into a form with which the associated equations of motion can be interpreted as a free center motion plus a decoupled one particle motion under an external influence. The new canonical variables are given by

$$X^\mu = \frac{1}{\sqrt{2}}(x_a^\mu + x_b^\mu) , \quad x^\mu = \frac{1}{\sqrt{2}}(x_a^\mu - x_b^\mu) , \quad (37)$$

and the corresponding P_μ and p_μ . We have $\mathcal{H} = \frac{1}{2}P_\mu P^\mu + \frac{1}{2}p_\mu p^\mu + \Phi(\sqrt{2}|x^\mu|)$. The (X^μ, P_μ) set describes free motion of (invariant) mass M given by $P_\mu P^\mu = -M^2 c^2$. The other set

(x^μ, p_μ) describes a, under the abuse of terminology, particle motion with σ -Hamiltonian $\mathcal{H}_r = \frac{1}{2}p_\mu p^\mu + \Phi(\sqrt{2}|x^\mu|)$, in which $\Phi(\sqrt{2}|x^\mu|)$ can be interpreted as an external potential. In particular

$$\frac{dp_\mu}{d\sigma} = -\frac{\partial \mathcal{H}_r}{\partial x^\mu} = -\frac{\partial \Phi}{\partial x^\mu} . \quad (38)$$

It is obvious that at the limit of vanishing Φ , we do retrieve two free particles. We also have \mathcal{H}_r being constant. However, with nonzero Φ , $p_\mu p^\mu$ will not be constant and the idea of a constant m being essentially the magnitude for the momentum p^μ cannot apply. Nor in the same sense rest masses for particles a and b .

To take the first example of a ‘nontrivial’ interaction, we consider the case of a collision, *i.e.*, an interaction completely localized in space-time. Before and after the collision, we have two free particles and constant momenta p_a^μ and p_b^μ and $p_a'^\mu$ and $p_b'^\mu$, respectively. Notice that our four-momenta in the problem behave mathematically the same as those of the three-momenta in Galilean/Newtonian mechanics. The collisional interaction is supposed to be nontrivial only at the zero of x^μ . Otherwise, we have $\frac{dp_\mu}{d\sigma} = 0$; in fact, we have $\frac{dp_a^\mu}{d\sigma} = \frac{dp_b^\mu}{d\sigma} = 0$. Before collision, at $\sigma < \sigma_o$, one requires

$$\begin{aligned} x_a^\mu &= p_a^\mu(\sigma - \sigma_o) + x_o^\mu , \\ x_b^\mu &= p_b^\mu(\sigma - \sigma_o) + x_o^\mu , \\ x^\mu &= (p_a^\mu - p_b^\mu)(\sigma - \sigma_o) . \end{aligned} \quad (39)$$

And we have similar picture after the collision. With total four-momentum (P_μ) conservation, one can have only $(p_a'^\mu, p_b'^\mu) = (p_b^\mu, p_a^\mu)$, as the value of σ -Hamiltonian is also constant. So we have a two to two scattering, with in fact the mass values interchanged after the collision (at $\sigma > \sigma_o$). At the center of four-momentum frame, in particular, we have $P_\mu = 0$ or $p_a^\mu = -p_b^\mu$. The two particles have equal and opposite four-momenta hence one has to have a negative energy value. Again, one is to be interpreted as an antiparticle in the t -picture. The story is actually more complicated than that. As illustrated in Fig.2, tracing the time evolution in the t -picture, instead of a particle with p_a^μ colliding/scattering with one with p_b^μ to give a complete four-momentum exchange, we have one with p_a^μ (assuming $p_a > 0$) and another with $p_a'^\mu$ but interpreted as an antiparticle having positive physical energy and four-momentum $-p_a'^\mu (= -p_b^\mu)$ colliding/scattering to give a particle with $p_b'^\mu = p_a^\mu$ and a antiparticle with $-p_b^\mu$. Scattering of particles a and b in the σ -picture becomes scattering of particles a of $\sigma < \sigma_o$ and a of $\sigma > \sigma_o$ into particles b of $\sigma < \sigma_o$ and b of $\sigma > \sigma_o$. The

idea of identity of particles is not the same in the two pictures. At the center of momentum frame in the σ -picture, we have the t -picture scattering of a particle with an antiparticle, but with identical four-momenta as $-p'_a{}^\mu = -p_b{}^\mu = p_a{}^\mu$. Mathematically, eliminating σ gives for particle a of $\sigma < \sigma_o$ and $\sigma > \sigma_o$

$$x_a^i = \frac{cp_a^i}{p_a^0}(t_a - t_o) + x_o^i \quad \text{and} \quad x_a^i = \frac{cp_a^i}{p_a^0}(t'_a - t_o) + x_o^i ,$$

with $t_a, t'_a > t_o$. At the center of four-momentum frame, $\frac{cp_a^i}{p_a^0} = \frac{cp_a^i}{p_a^0}$ as a common negative sign shows up in both the numerator and denominator. That gives $x_a^i = x_a^i$ at any value of time. So, the particle and antiparticle are always at the same position in space, but have different σ values which change in opposite directions. At time t_o , the σ values become the same and they interact and switch their four-momenta. The particles masses are of course the same as a result of the choice of frame of reference. Particle rest mass is reference dependent under momentum boosts. One can consider however the two particles a and b to have different identities characterized by some conserved charges, like one being an electron and the other a quark. The t -picture story is then essentially one of electron-positron annihilation creating a quark-antiquark pair. The particle-antiparticle at the same position and two pairs of the same masses part maybe consdier an artefact of the choice of reference frame. For example, taking a center of energy frame with $p_a^i = p_b^i$ for the two Poincaré-Snyder particles will give the t -picture of particle-antiparticle moving towards or away from one another. Notice that a momentum boost frame transformation transforms also the potential Φ in general.

The collision/scattering story looks interesting and quite successful. We have provided, for instance, a description of something like particle-antiparticle creation and annihilation in a classical particle dynamics setting in our σ -mechanics. That is a nontrivial success of the formulation. Further investigations of σ -mechanics with nontrivial interactions may provide a lot more in the future. We note in passing the as the σ -Hamiltonian for a tachyon is positive, the σ -Hamiltonian for a system of a particle and a tachyon can be zero. It is possible for such a pair to annihilate into nothing — a truly intriguing scenario.

Here in this paper, we want to focus mostly on providing and justifying a formulation of σ -mechanics — the direct canonical formulation presented. We will stop here on analysis of specific σ -Hamiltonian. Next, we look briefly at the corresponding quantum mechanics.

III. QUANTIZATION IN PERSPECTIVE

Independent of its role as an intermediate framework between the elusive Quantum Relativity and the familiar relativities [1], one argument in favor of replacing the Poincaré symmetry by the Poincaré-Snyder $G(1, 3)$ is the more natural behavior under quantization with space-time position operator X^μ contained in the symmetry algebra [1, 11]. We have presented an explicit description of the quantization of free particle σ -mechanics under the geometric perspective of the $U(1)$ central extension of $G(1, 3)$ [1], following closely the approach of Refs.[9, 10]. At the classical level, the σ -Lagrangian $\mathcal{L} = \frac{1}{2}p^\mu p_\mu$ is only quasi-invariant under the momentum boosts $\dot{x}^\mu \rightarrow \dot{x}^\mu + \mathcal{P}^\mu$ ($\dot{x}^\mu = p^\mu$) :

$$\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + \frac{d}{d\sigma}\eta_{\mu\nu} \left(\frac{1}{2}\mathcal{P}^\mu \mathcal{P}^\nu \sigma - x^\mu \mathcal{P}^\nu \right) \equiv \mathcal{L} + \frac{d}{d\sigma}\Delta(\sigma, x^\mu; \mathcal{P}^\mu) . \quad (40)$$

When compared to the Galilean case, the only difference is the missing of a parameter corresponds to the central charge (which is the particle mass in the Galilean case). Note that the Δ term though having no effect on the equations of motion, does affect the definition of the Noether charges. Geometrically, the nontrivial cohomology of the $G(1, 3)$ group gives rise to the nontrivial $U(1)$ central extension which represents the phase transformation in the quantum description of a state. The feature is missing in the usual Einstein relativistic quantum mechanics with trivial group cohomology.

On the quantum level, we have the central extension given by

$$[K'_\nu, P_\mu] = i\hbar \eta_{\mu\nu} F ,$$

where we have re-scaled the algebra to put in the $i\hbar$ for the quantum case [*cf.* Eq.(2)]. The generators of the momentum boosts K'_μ have the properties of covariant space-time position operator, as illustrated in Ref.[1]. From the analysis of the Quantum Relativity level [2], we actually expect noncommuting $X_\mu = \frac{1}{\kappa c}K'_\mu$. That perspective reinforces the result that unlike the Galilean $X_i = \frac{1}{m}K_i$, K'_ν is essentially the position operator for all ‘particle’ state independent of any ‘particle’ properties like m . Again, m is not an invariant under the momentum boosts. The commutator of the central extension is to be identified directly as the Heisenberg commutation relation. We have hence a consistent result, namely to arrive at quantum mechanics as we know it, the central charge has to be taken as unity instead of a free parameter. The other commutator relation re-casted as

$$P_\mu = \frac{dK'_\mu}{d\sigma} = \frac{1}{i\hbar}[K'_\mu, H'] \quad (41)$$

gives the Heisenberg equation of motion.

Next, we write down basic results from the more common quantization framework, namely a canonical quantization based on the Schrödinger picture.

The Schrödinger equation for σ -mechanics, or called the σ -dependent covariant Schrödinger equation,

$$i\hbar \frac{d}{d\sigma} |\psi, \sigma\rangle = \hat{\mathcal{H}} |\psi, \sigma\rangle \quad (42)$$

is obtained in Ref.[1] from the group/geometric analysis on the free particle case. The operator representation for the σ -Hamiltonian $\hat{\mathcal{H}} \rightarrow i\hbar \frac{d}{d\sigma}$ on the Hilbert space of quantum states $|\psi, \sigma\rangle$ is the starting point for the standard canonical quantization, in accordance with the classical canonical formalism above. That is, $\mathcal{H}(x^\mu, p_\mu) \rightarrow \hat{\mathcal{H}}(\hat{x}^\mu, \hat{p}_\mu)$. The operators \hat{x}^μ and \hat{p}_μ satisfy the fundamental canonical commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = 0, \quad [\hat{p}_\mu, \hat{p}_\nu] = 0, \quad [\hat{x}^\mu, \hat{p}_\nu] = i\hbar \delta^\mu_\nu. \quad (43)$$

At least formally, an x -representation can be written by taking

$$\hat{x}^\mu \rightarrow x^\mu, \quad \hat{p}_\mu \rightarrow -i\hbar \frac{\partial}{\partial x^\mu},$$

giving the generic particle Hamiltonian $\hat{\mathcal{H}}(\hat{x}^\mu, \hat{p}_\mu) = \frac{-\hbar^2}{2} \partial^\mu \partial_\mu + \Phi(x^\mu, \sigma)$. Explicitly, the Schrödinger equation becomes

$$i\hbar \frac{\partial \psi(x^\mu, \sigma)}{\partial \sigma} = \left[\frac{-\hbar^2}{2} \partial^\mu \partial_\mu + \Phi(x^\mu, \sigma) \right] \psi(x^\mu, \sigma), \quad (44)$$

where $\psi(x^\mu, \sigma)$ is the space-time wavefunction for the state at σ . In the case that the ‘potential’ does not depend on σ explicitly, *i.e.* $\Phi = \Phi(x^\mu)$, we can perform separation of variables and obtain a generalized Klein-Gordon equation:

$$\begin{aligned} \psi(x^\mu, \sigma) &= \phi(x^\mu) \Sigma(\sigma) \\ \Rightarrow \quad i\hbar \frac{\partial \Sigma(\sigma)}{\partial \sigma} &= -\frac{1}{2} m^2 c^2 \Sigma(\sigma) \\ \text{and} \quad \partial^\mu \partial_\mu \phi(x^\mu) - \frac{m^2 c^2}{\hbar^2} \phi(x^\mu) &= \frac{2}{\hbar^2} \Phi(x^\mu), \end{aligned} \quad (45)$$

where we have, following the analysis of the classical mechanics, written the eigenvalue of the σ -Hamiltonian as $-\frac{1}{2}m^2c^2$. Of course for $\Phi = 0$, one obtains the Klein-Gordon equation for Einstein relativistic mechanics of a free (spin 0) particle.

Taking the Schrödinger equation on an abstract state to that of the x -representation can also be performed in the standard fashion. One assume a complete set of orthonormal space-time position eigenstates $\hat{x}^\mu|\mathbf{x}\rangle = x^\mu|\mathbf{x}\rangle$ satisfying

$$\langle\mathbf{x}'|\mathbf{x}\rangle = \delta^4(\mathbf{x}' - \mathbf{x}) \quad \text{and} \quad \int d^4\mathbf{x} |\mathbf{x}\rangle\langle\mathbf{x}| = \mathbf{1} .$$

The wavefunction $\psi(x^\mu, \sigma)$ has values given by the expansion coefficients $\langle\mathbf{x}|\psi, \sigma\rangle$ in

$$|\psi, \sigma\rangle = \int d^4x |\mathbf{x}\rangle\langle\mathbf{x}|\psi, \sigma\rangle .$$

A question arise as to the interpretation of the wavefunction. A naive Born interpretation looks like possible and at the same time problematic. It is easy to see that the integral of wavefunction magnitude-squared can natural give the relative probability of finding the quantum state to be within the space-time region. However, the normalization as unit probability sounds unconventional. Similar comment applies to the the expectation value of an appropriately defined operator over a particular state. Here, we would rather if the question open for the moment, awaiting further analyses of various system Hamiltonians to give more insight for addressing the question.

We will give a coherent state representation *a la* Klauder[17] in a forth-coming paper[18] in which we also discuss the case of harmonic oscillator under Poincaré-Snyder relativity as well as the path-integral picture.

IV. CONCLUSIONS

We introduced the Poincaré-Snyder relativity as an extended version of Einstein relativity with $G(1, 3)$ symmetry [1]. It is a contraction limit of the full Quantum Relativity. The relativity has in addition to the Poincaré $ISO(1, 3)$ transformations an extra class of transformations called momentum boosts [6], dependent on the new parameter σ . We want to be exceptionally cautious before committing to a particular physical picture about the parameter and the transformations. So, we take a minimalist approach here, trying to what could be the Poincaré-Snyder mechanics and if and how we could make sense out of it before even understanding about the physics of the σ parameter and the transformations.

We show here there is a straightforward canonical formulation of classical and quantum mechanics under the Poincaré-Snyder relativity, with σ as a formal ‘evolution’ parameter.

The formulation gives free particle mechanics essentially the same as Einstein relativity though without the rest mass as an intrinsic defining properties of the particle. And a Poincaré-Snyder can be a particle, antiparticle, tachyon, and tachyonic antiparticle, under the time evolution (t -picture) of Einstein relativity. So long as a human observer is concerned it is clear that it is the t -picture that can describe our laboratory experience. This is the wisdom from Stückelberg[14]. The interpretation is achieved within a particular reference frame relative to the momentum boosts. As we have no prior experience or do not know about momentum boosts transformations in experiments so far, it is in fact natural to expect that we have been studying physics essentially or approximately in one specific momentum boost frame.

To go beyond the free particle case, we look at the case with an insurmountable potential barrier in the time direction. The ‘rebound’ turns a particle into an antiparticle. The two particle collision is the next setting, and the first case of nontrivial interaction we analyze here. All the results are dictated directly by the mathematics within the canonical picture, and looks interesting and quite successful. We have provided, for instance, a direct description of something like particle-antiparticle creation and annihilation in a classical particle dynamics setting in our σ -mechanics. That is a nontrivial success of the formulation.

Quantization of the canonical formulation is also straightforward. It matches exactly to the direct group geometric quantization picture based on the $U(1)$ central extension of $G(1,3)$ [1]. The latter illustrates that $G(1,3)$ has the superior properties for giving rise to the conventional relation between canonical realization as in classical mechanics and projective representation as in quantum mechanics — one the the $ISO(1,3)$ symmetry fails. The potential to use our Poincaré-Snyder formulation to re-analyze every aspects of relativistic quantum mechanics including foundation issues, information theoretical aspects, and various applications is unlimited. It is likely to provide, among other things, a better picture to describe macroscopic superposition of states localized otherwise at different space-time points [19].

What we have taken here is an important step towards understanding and building the dynamics for our Quantum Relativity, and plausibly the necessary step to find direct connection with the experimental front at the more familiar scale (rather than Planck scale). The formulation of Poincaré-Snyder mechanics here gives us good confidence that we are on the right track. And the theory is of interest on its own. It could provides a new perspective

and mathematical setting to handle the usual (Einstein) relativistic mechanics both at the classical and quantum levels. It would also describe a much boarder class of phenomena to be studied experimentally.

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FIG. 1: Particle-antiparticle annihilation from the time-evolution picture given as a Poincaré-Snyder particle rebounds from an insurmountable potential barrier in the time direction. Note that after the rebound, at $\sigma > \sigma_o$, the free motion has negative slope in the t Vs σ plot, hence negative energy. It gives an antiparticle as interpreted in the Einstein relativity time evolution picture (t -picture). t -picture evolution is to be read from the σ -axis upward, as versus the formal σ -picture evolution going left to right. The single Poincaré-Snyder particle is seen in the t -picture as a pair of particle and antiparticle for $t < t_o$ and nothing for $t > t_o$; they are annihilated at $t = t_o$. We also give the matching x Vs σ plot, with the t -picture reading indicated by the short (black) arrows.

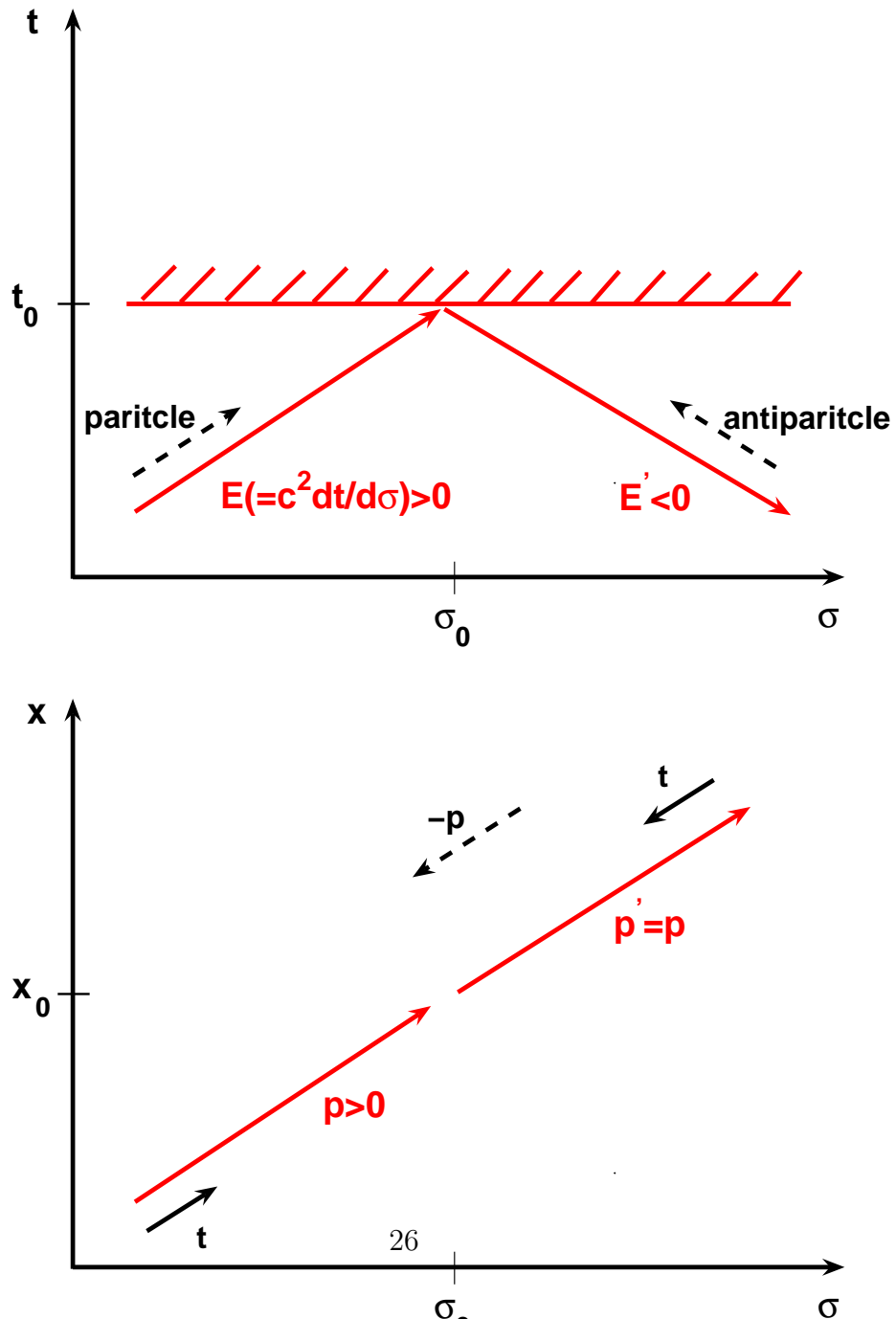


FIG. 2: Collision/scattering of two Poincaré-Snyder particles a and b illustrated, in the center of energy frame. This is a t Vs σ plot, slope of which give the time component of the particle four-momentum, essentially energy. Negative slope gives a antiparticle as interpreted in the Einstein relativity time evolution picture (t -picture). t -picture evolution is to be read from the σ -axis upward, as versus the formal σ -picture evolution going left to right. Note that a and b label particle identity in the σ -picture, which are not characterized by their masses. And the picture does not trace the same Einstein particle identity as in the t -picture.

